The Kondo Effect and Conformal Field Theory

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1 The Kondo Effect and RG

It is an experimentally known fact that classes of metals with magnetic impurities exhibit a minimum resistivity at a certain temperature T_K, below which the resistivity begins to rise. The existence of this minimum is, on its face, paradoxical, as standard phonon theory in metals predicts that the resistivity should be a monotonically increasing function of temperature. Moreover, this minimum temperature does not depend on the *concentration* of the impurities, so it is not due to electronic correlation between neighboring impurities, but must rather be due to the scattering effect between the electrons and the magnetic impurity. Kondo proposed a model in 1964 that reproduced this effect, and so it has since been known as the *Kondo effect*. It can be described be the following Hamiltonian:

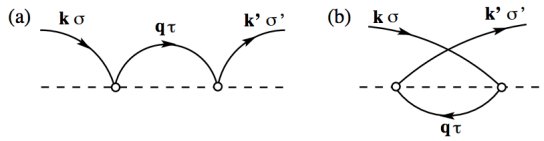
$$H = \sum_{k,\alpha} \psi^{\dagger\alpha}(k)\psi_\alpha(k)\epsilon(k) + \lambda \vec{S} \cdot \sum_{k,k'} \psi^{\dagger\alpha}(k)\frac{\vec{\sigma}^\beta_\alpha}{2}\psi_\beta(k) \tag{1}$$

The simple Drude model of conductance gives a resistivity in the presence of a single impurity as:

$$\rho(T) = \frac{m}{ne^2\tau_{Drude}} = \frac{m}{ne^2}\Gamma(\epsilon_F).$$

Here τ_{Drude} is the single-particle transport life-time. It is related to the inverse of the scattering rate Γ for an electron at the fermi energy ϵ_F. Since the scattering rate is proportional to angular integrate matrix element of the T matrix, we see that the scattering matrix determines the resistivity.

The Kondo effect comes from the renormalization of the propagator. The contributing diagrams involve spin flips, as is explained in detail in [1]. The contributing Feynman diagrams are below.



Here, τ is the opposite spin from σ. From integrating out the high-energy modes, we obtain a beta function of $\beta(\lambda) = -\nu\lambda^2$ where ν is the density of states. This gives us that λ runs as

$$\lambda_{eff} = \frac{\lambda_0}{\log(T/T_K)} \tag{2}$$

Here D is the UV cutoff for the bandwidth: $|k - k_F| < D$. This gives T_K to be on the order of $De^{-1/\nu\lambda_0}$. This calculation was done by Kondo, giving the correct temperature at which perturbation theory breaks down. To be able to describe the full *non-interacting* picture, we need the full mechanics of Wilson's numerical renormalization scheme. However, from the more modern perspective of understanding field theories as flows from UV to IR fixed points, we can develop our understanding of the Kondo problem (and even calculate relvant low-temperature quantities) in terms of the 2D conformal field theory at $T = 0$.

2 Reduction to 1+1 Dimensions

The following 4 sections will summarize the analysis and results of [2]. Note that near the Fermi energy $\epsilon(k) \approx v_F(k-k_F)$. Since we are looking at only s-wave scattering, consider the equivalent one dimensional problem.

$$\begin{aligned}\psi(k) &\to \frac{1}{\sqrt{4\pi k}}\psi_0(k)\\ H_0 &= v_F\int dk(k-k_F)\psi_0^\dagger(k)\psi_0(k)\\ &= v_F\int dk k\psi_0^\dagger(k+k_F)\psi_0(k+k_F)\\ H_{int} &= \lambda v_F\nu\int dkdk'\psi_0^\dagger(k)\frac{\vec{\sigma}}{2}\psi_0(k)\end{aligned}$$

We can then write $\psi_L = \int dke^{ikr}\psi_0(k+k_F)$ for an incoming wave and $\psi_R = \int dke^{-ikr}\psi_0(k+k_F)$ for an outgoing wave. These sum together to give the full position space Fourier transform of ψ. We then get:

$$\begin{aligned}H_0 &= \frac{v_F}{2\pi}\int_0^\infty dr\left(\psi_L^\dagger(r)i\frac{\partial}{\partial r}\psi_L(r) - \psi_R^\dagger(r)i\frac{\partial}{\partial r}\psi_R(r)\right)\\ H_{int} &= \lambda v_F\nu\int dkdk'\psi_0^\dagger(k)\frac{\vec{\sigma}}{2}\psi_0(k) = v_F\nu\lambda\,\vec{S}\cdot\psi^\dagger(0)\frac{\vec{\sigma}}{2}\psi(0)\end{aligned} \tag{3}$$

Since $\psi_{L,R}(r,t)$ are defined only for r positive, we can define

$$\begin{aligned}\psi_L(z=\tau+ix) &= \psi_L(x,\tau)\\ \psi_R(z^*=\tau-ix) &= \psi_R(x,\tau).\end{aligned} \tag{4}$$

for τ imaginary time. Further, because these must agree at $x=0$, we can write everything in terms of ψ_L with $\psi_L(-x,\tau) := \psi_R(x,\tau)$. Then we can write

$$\mathcal{H}_0 = \frac{v_f}{2\pi}\psi_L^\dagger i\frac{d}{dx}\psi_L. \tag{5}$$

This 1+1D free fermion theory is in fact conformally-invariant (i.e. a CFT). The two-point functions can easily be computed as:

$$\langle\psi_L(z)\psi_L^\dagger(0)\rangle = \frac{1}{v_F}\frac{1}{z},\quad \langle\psi_R(z^*)\psi_R^\dagger(0)\rangle = \frac{1}{v_F}\frac{1}{z^*}$$

We will set $v_F = 1$ and rescale $\nu\lambda \to \lambda$ for future discussion. We can also now write the full 1+1D Hamiltonian as a lattice model as:

$$H = t\sum_i(\psi_i^\dagger\psi_{i+1} + \psi_{i+1}^\dagger\psi_i) + \lambda\,\vec{S}\cdot\psi_0^\dagger\frac{\vec{\sigma}}{2}\psi_0 \tag{6}$$

At $\lambda = 0$, this free fermion model has even and odd parity wave functions going as $\cos kx$ and $\sin kx$, respectively. On the other hand, if we make the interaction term large $\lambda \gg t$, the electron at the origin will want to do everything possible to ensure that the state at the origin is a singlet. Thus, the ground state will be a singlet at 0, and fluctuations away from 0 are allowed, so long as particles and holes do not interfere with the singlet state at the origin.

3 The role of CFT

The CFT side of the picture emerges from the free electron model by defining the current density operator in terms of the normal order of the ψ field.

$$J(x-t) =: \psi_L^\dagger \psi_L :\equiv \lim_{\epsilon\to 0} \left(\psi_L(x)\psi_L(x+\epsilon) - \langle 0|\, \psi_L^\dagger(x)\psi_L(x+\epsilon)\,|0\rangle\right). \tag{7}$$

This in turn satisfies an OPE:

$$: J(x)J(x+\epsilon) := 2i\, :\psi^\dagger(x) i\frac{d}{dx}\psi(x): \tag{8}$$

But this is just the free theory Hamiltonian, so (up to an infinite constant), we get:

$$\mathcal{H} = \frac{1}{4\pi} J^2. \tag{9}$$

It is here, in this surprisingly trivial setting, that we begin to see an affine algebra structure begin to emerge.

If we add spin indices to ψ, we get two current densities, corresponding to charge and spin:

$$\begin{aligned} J &=: \psi^{\dagger\alpha}\psi_\alpha :, \qquad \vec{J} =: \psi^{\dagger\alpha} \frac{\vec{\sigma}_\alpha^\beta}{\psi}_\beta : \\ : J^2 : &=: \psi^{\dagger\alpha}\psi_\alpha \psi^{\dagger\beta}\psi_\beta : + 2i\psi^{\dagger\alpha}\frac{d}{dx}\psi_\alpha \\ : \vec{J}^2 : &= -\frac{3}{4} :\psi^{\dagger\alpha}\psi_\alpha\psi^{\dagger\beta}\psi_\beta : + \frac{3i}{2}\psi^{\dagger\alpha}\frac{d}{dx}\psi_\alpha \end{aligned} \tag{10}$$

These can be directly seen to satisfy commutation relations:

$$\begin{aligned} [J(x), J(y)] &= 4\pi i \frac{d}{dx}\delta(x-y) \\ [J^a(x), J^b(y)] &= 2\pi i \epsilon^{abc} J^c(x)\delta(x-y) + \pi i \delta^{ab}\frac{d}{dx}\delta(x-y) \\ [J(x), \vec{J}(y)] &= 0 \\ \Rightarrow \mathcal{H} &= \underbrace{\frac{1}{8\pi}J^2}_{\mathcal{H}_c} + \underbrace{\frac{1}{6\pi}\vec{J}^2}_{\mathcal{H}_s}. \end{aligned} \tag{11}$$

We thus can express the Hamiltonian in terms of two commuting conserved charges. This is the important point, as it allows us to decouple the "charge" from the "spin" degrees of freedom. We write $H = H_c + H_s$.

In terms of Fourier modes $J_n^a \equiv \frac{1}{2\pi}\int_{-l}^{l} e^{in\pi x/l} J^a(x)$, this gives:

$$[J_n^a, J_m^b] = i\epsilon^{abc} J_{n+m}^c + \frac{1}{2} n \delta_{n,-m}. \tag{12}$$

This is the affine $SU(2)$ Kac-Moody algebra of level[1] $k=1$, denoted $\widehat{SU(2)}_1$. Note that this gives

$$H_s = \frac{\pi}{3l}\sum_{n=-\infty}^{\infty} :\vec{J}_{-n}\vec{J}_n: . \tag{13}$$

[1] For some reason some literature calls k the central charge, which is not consistent with other more usual definitions of the true Virasoro central charge c.

4 Boundary Conditions and Strong-Weak Fixed Points

In the last section, we expressed the free fermion theory in terms of a $SU(2)$ Kac-Moody algebra at $k = 1$. The boundary conditions on the free theory at $x = \pm l$ are that $\psi(l) = -\psi(-l)$. This gives a spectrum

$$k = \frac{\pi}{l}\left((n_\uparrow + \frac{1}{2}) + (n_\downarrow + \frac{1}{2})\right) \tag{14}$$

For a charge $Q = Q_\uparrow + Q_\downarrow$ state we get an energy of:

$$E = \frac{\pi V}{l}[\frac{1}{2}Q_\uparrow^2 + \frac{1}{2}Q_\downarrow + \sum_{m=-\infty}^{\infty} m(n_m^\uparrow + n_m^\downarrow)], \tag{15}$$

where the last term corresponds to excitations gained from raising n_m electrons up m levels, and is zero for the ground state. The boundary conditions determine the spectrum and couple the Q and S^z quantum numbers so that

$$Q = 2S^z \text{ mod } 2. \tag{16}$$

This means that, in terms of representations of the KM algebra, the states must be obtained by applying appropriate raising operators (J^a_{-n}) to the ground states that are either even charge integer spin or odd charge half-integer spin. The total Hilbert space is the sum of two irreducible representations of $\widehat{SU(2)}_1$ (known as "conformal towers" in the literature):

$$(\text{even}, \text{integer}) \oplus (\text{odd}, \text{half-integer}).$$

It is quick to see that if we have the opposite boundary conditions, $\psi(l) = \psi(-l)$, corresponding to the *phase-shifted* case at the interacting fixed point, we'd get the reverse boundary conditions:

$$(\text{even}, \text{half-integer}) \oplus (\text{odd}, \text{integer}).$$

The fact that the interacting fixed point can be written as a free fermion theory with different boundary conditions leads one to consider whether we can obtain *another* Kac-Moody algebra in the perturbed hamiltonian:

$$\mathcal{H} = \psi^{\dagger\alpha} i\frac{d}{dx}\psi_\alpha + \lambda\psi^{\dagger\alpha}\frac{\vec{\sigma}_\alpha^\beta}{2}\psi_\beta \cdot \vec{S}\,\delta(x) = \frac{1}{8\pi}J^2 + \frac{1}{6\pi}\vec{J}^2 + \lambda\vec{J}\cdot\vec{S}\,\delta(x) \tag{17}$$

This is just a perturbation of the spin Hamiltonian, which can be written in Fourier space as:

$$H_s = \frac{\pi V}{l}\left(\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\vec{J}_{-n}\cdot\vec{J}_n + \lambda\vec{J}_n\cdot\vec{S}\right)\right)$$

It looks like at $\lambda = 2/3$, this factors into (up to an irrelevant constant):

$$H = \frac{\pi V}{3l}\sum_{n=-\infty}^{\infty}(\vec{J}_{-n} + \vec{S})\cdot(\vec{J}_n + \vec{S})$$

Defining $\vec{\mathcal{J}}_n = \vec{J}_n + \vec{S}$, we get an algebra satisfying the *exact* same commutation relations, but with $\vec{\mathcal{J}}$ instead of $\vec{J}$. This is again a KM algebra, meaning it is a CFT which we posit describes the *strong-coupling fixed point*. Note that because the total spin operator has been shifted by a factor of one half, we have changed "integer" $\rightarrow$ "half-integer" in the conformal towers above, consistent with changing boundary conditions.

From here, we can use the correlation functions in this CFT to calculate low temperature quantities as well.

5 Multi-Channel Kondo Effect

If we have multiple "channels" for the different electrons (e.g. different f and d shell orbitals), the Hamiltonian could be written initially as the following form:

$$H = \sum_{\vec{k},\alpha,i} \epsilon(k) \psi_k^{\dagger\alpha,i} \psi_{k,\alpha,i} + \lambda \vec{S} \cdot \sum_{\vec{k},\vec{k}',\alpha,\beta,i} \psi_{\vec{k}}^{\dagger\alpha,i} \frac{\vec{\sigma}_\alpha^\beta}{2} \psi_{\vec{k}',\beta,i} \tag{18}$$

where i runs from $1 \to k$, and the model has $SU(k)$ symmetry. We will here let the impurity have an arbitrary spin S, not necessarily equal to $1/2$.

It will now be interesting to see whether these multi-channel results could be recreated using the Kondo effect in quantum dots. In order to make accurate predictions, we will need to understand the symmetry breaking of the $SU(k)$. For an initial experimental realization, see [3].

The previous discussion of KM algebras immediately generalizes. We now have to introduce an additional *flavor* current $J_A = \psi$ with commutation relations for the Fourier modes given as:

$$[J_n^A, J_M^B] = i f^{ABC} J_{n+m}^C + n \delta_{n,-m}^{AB} \tag{19}$$

where f^{ABC} are the structure constants for $SU(k)$. This is the KM algebra for $SU(k)$ with central charge 2. The level of the $SU(2)$ spin algebra gets modified (because there are now k flavors) and becomes k instead of 1. The central charge[2] charge of such an algebra is known to be:

$$c_{G,k} = \frac{k \dim G}{\tilde{h}_G + k} \tag{20}$$

Where $\tilde{h}_G$ is the dual Coxeter number which is k for $SU(k)$. A quick check gives that the total central charge of the CFT is k, consistent with there being $2k$ species of free fermions. The free Hamiltonian is quickly calculated to be:

$$\mathcal{H} = \frac{1}{8\pi k} J^2 + \frac{1}{2\pi(k+2)} \vec{J}^2 + \frac{1}{2\pi(k+2)} (J^A)^2. \tag{21}$$

As before, we now add the Kondo term to the Hamiltonian of the free theory. It only modifies the spin part as:

$$\mathcal{H}_s = \frac{1}{2\pi(k+2)} \vec{J}^2 + \lambda \vec{J} \cdot \vec{S}\, \delta(x). \tag{22}$$

Again, we see a special value of $\lambda = \frac{2}{2+k}$ where this factors.

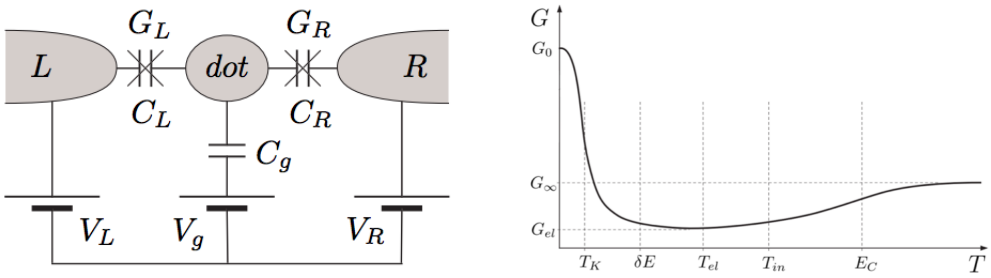
The conformal towers of $\widehat{SU(2)}_k$ have been worked out to be one of $k+1$ possible irreps labeled by a lowest energy (highest weight) state $s = 0, 1/2, 1, \ldots, k/2$ from which the Hilbert space is again obtained by acting on it by $\vec{J}_{-n}$. To understand which irreps appear, we need to know their relationship to boundary conditions.

Here, it is necessary to understand the general relationship between the constrains of conformally-invariant boundary conditions and the fusion rules of WZW-type models involving KM algebras. The essential point is this: The low-temperature fixed point of the strongly-interacting theory with a spin s at the center is given by fusion of the free theory with a spin s primary for $s \leq k/2$ and a spin $k/2$ primary for $s > k/2$. This can be understood as tensoring with the module for that primary.

[2] It is interesting to note that this is proportional to the specific heat. I will investigate this further at a later point.

6 Relationship to Quantum Dots

Quantum dots allow for a tunable Kondo effect [4], making them an excellent candidate to build experimental realizations of the physics discussed above. In the quantum dot case, the magnetic impurity given by coulomb blockaded electrons in a quantum dot makes it so that the *conductance* rather than the resistivity is proportional to scattering. We have the following figures from [5], describing the effective circuit element of the quantum dot and the conductance curve:



At large temperatures, the dot has nearly a continuum of states, and the total conductivity is given by the reciprocal sum of the of the conductivities of the left and right contacts of the dot:

$$\frac{1}{G_\infty} = \frac{1}{G_L} + \frac{1}{G_R} \tag{23}$$

Once T goes below the scale of the coulomb charging energy but is still larger than the level spacing in the quantum dot, $\delta_E \ll T \ll E_C$ fermi statistics become relevant for the conductivity, and we get:

$$G = G_\infty \frac{E_C(N - N^*)/T}{\sinh 2E_C(N - N^*)/T} \tag{24}$$

Here N is a dimensionless parameter proportional to the gate voltage $C_g V_g/e$. For N a half integer, we have a coulomb blockade peak at $N = N^*$.

At lower temperatures, still above δ_E, virtual transitions dominate, and we observe the increase of elastic co-tunneling. Below the level spacing δE, however, the single unpaired electron on the dot (placed there via appropriate choice of gate voltage) can no longer scatter on the other spins in the dot. This single unpaired $s = 1/2$ spin gives rise to the Kondo effect. As before, at $T = 0$ it will be completely screened, forming a singlet state.

The 1D picture developed in Section 2 applies just as well to this quantum dot circuit. It is interesting to consider experimental realizations of (possibly broken) flavor symmetry arising from multiple channels. It is also worth thinking about possibly forming a higher spin (e.g. triplet) state in a double quantum dot (usually taken to be two quantum dots in series). Further calculations can explore different circuit configurations of quantum dots.

It is worthwhile to investigate how these KM CFTs might generalize to characterize systems incorporating spin-orbit coupling.

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